$$Y' = A(t)Y + G(t)$$

$Y_c(t)$ = complementary solution : $Y_c' = A(t)Y_c$

$Y_p(t)$ = particular solution : $Y_p' = A(t)Y_p + G(t)$

Review L17-18(4.8, 6.2, 6.4-6.6)

4.8 Nonhomogeneous Linear Systems

$$\mathbf{y}'(t) = A(t)\mathbf{y}(t) + \mathbf{g}(t)$$

(1) **General Solution: $\mathbf{y}(t) = \mathbf{y}_c(t) + \mathbf{y}_p(t)$**

(2) **Two Methods of finding y_p:**

- The Method of Undetermined Coefficients: Find the form of $\mathbf{y}_p$ from $\mathbf{g}(t)$
- The Method of Variation of Parameters

1. The Method of Undetermined Coefficients:

The method of undetermined coefficients to find a particular solution $\mathbf{y}_p$ to the nonhomogeneous linear system

$$\mathbf{y}'(t) = A\mathbf{y}(t) + \mathbf{g}(t)$$

where

(1) the coefficient matrix A is an $n \times n$ **constant matrix**, and

(2) the entries of $\mathbf{g}(t)$ are

(a) Polynomials,

(b) Exponential functions($e^{\alpha t}$),

(c) Sine/Cosine functions($\cos(\alpha t)/\sin(\alpha t)$), or finite sums and products of these functions.

For example,

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{y} + \mathbf{g}(t) \quad \begin{cases} y_1' = y_2 + g_1 \\ y_2' = y_1 + g_2 \end{cases}$$

① $Y_c : (\lambda = 1, \begin{bmatrix} 1 \\ 1 \end{bmatrix})(\lambda = -1, \begin{bmatrix} -1 \\ 1 \end{bmatrix})$

$Y_c = C_1 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

② Let $G(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}$

$Y_p = ?$ $\begin{bmatrix} a_1 t + b_1 \\ a_2 t + b_2 \end{bmatrix}$

$\mathbf{g}(t)$	$\mathbf{y}_p$
$\begin{bmatrix} t \\ 5 \end{bmatrix}$ polynomial	$y_p = \begin{bmatrix} a_1 t + b_1 \\ a_2 t + b_1 \end{bmatrix} = at + b$
$\begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}$ exponential	$y_p = \begin{bmatrix} a_1 e^{2t} \\ a_2 e^{2t} \end{bmatrix}$
$\begin{bmatrix} e^{2t} \\ t \end{bmatrix}$ exponential and polynomial	$y_p = \begin{bmatrix} a_1 e^{2t} + b_1 t + c_1 \\ a_2 e^{2t} + b_2 t + c_2 \end{bmatrix}$

Example: Find the general solution to the nonhomogeneous system

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \mathbf{y} + t \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

Complementary solution:

$$y' = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} y \qquad (\lambda = 4, \begin{bmatrix} 2 \\ 3 \end{bmatrix}) \ (\lambda = -1, \begin{bmatrix} -1 \\ 1 \end{bmatrix})$$

$$y_c = c_1 e^{4t} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Particular solution:

- Look for the particular solution $\mathbf{y}_p = t\mathbf{a} + \mathbf{b} = t \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$
- Substitute $\mathbf{y}_p'$ into the system: $\mathbf{y}_p' = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \mathbf{y}_p + t \begin{bmatrix} 2 \\ -4 \end{bmatrix}$

$$\overset{y_p'}{\cancel{\mathbf{a}}} = P(t\mathbf{a} + \mathbf{b}) + t \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

$$(ta + b)' = p(ta + b) + tg$$

$$a = t(pa + g) + pb$$

$$t\text{-term}: \quad pa + g = \vec{0} \qquad \underbrace{pa = -g}$$

$$\text{const}: \quad pb = a$$

$$\underset{P}{\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}} \underset{a}{\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}} = - \underset{-g}{\begin{bmatrix} 2 \\ -4 \end{bmatrix}} \Rightarrow \mathbf{a} = \begin{bmatrix} 3 \\ -\frac{5}{2} \end{bmatrix}$$

$$\overset{p}{\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}} \overset{b}{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}} = \overset{a}{\begin{bmatrix} 3 \\ -\frac{5}{2} \end{bmatrix}} \Rightarrow \mathbf{b} = \begin{bmatrix} -\frac{11}{4} \\ \frac{23}{8} \end{bmatrix}$$

Now we have

$$\mathbf{y}_p = t \begin{bmatrix} 3 \\ -\frac{5}{2} \end{bmatrix} + \begin{bmatrix} -\frac{11}{4} \\ \frac{23}{8} \end{bmatrix}$$

General solution: $\mathbf{y}(t) =$ $y_c + y_p$